Continuous-time Methods for Economics and Finance

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Introduction

Stochastic calculus was introduced in economics by Fischer Black, Myron Scholes and Robert C. Merton in the early 1970s.[1] This tool can make dynamic models significantly more tractable. The Black-Scholes option-pricing formula, for example, is significantly easier to handle than its discrete-time counterpart based on binomial trees. In macroeconomics, continuous-time techniques are increasingly becoming the standard in some important areas of research. For instance, since the 2007-9 financial crisis there is a growing body of literature analyzing the links between the financial sector and the economy as a whole. This 'macro-finance' literature typically employs continuous-time techniques to analyze issues such as the quantification of systemic risk or the propagation, due to nonlinear effects, of financial shocks to the rest of the economy.[2] In monetary policy analysis, continuous-time models offer tractable ways to analyze price-setting decisions with menu costs. In fiscal policy, models with endogenous default or heterogeneous-agents can be easily computed using stochastic calculus. Other applications can be found in growth theory, market microstructure or industrial organization.

With respect to standard discrete-time techniques, continuous-time methods have the following advantages:

- As discussed above, continuous-time methods often yield to analytical solutions. For example, dynamic programming problems such as the Merton optimal portfolio selection have closed-form solutions.[3] This is the workhorse model in macro-finance as it solves the problem of a

[1] Continuous-time calculus was developed in the 17^{th} Century by Isaac Newton and Gottfried Wilhelm Leibniz. Its extension to stochastic processes (stochastic calculus) is much more recent, after the pathbreaking work of Kiyoshi Itō in the 1940s and 1950s.

[2] See http://scholar.princeton.edu/markus/MacroMoneyFinance for references.

[3] See Merton, R. C. (1969). "Lifetime Portfolio Selection under Uncertainty: the Continuous-Time Case". *The Review of Economics and Statistics* 51 (3): 247–257.

risk-averse agent who consumes and saves in riskless and risky assets. For example, Adrian and Boyarchenko (2013) build on this to present a theory of financial intermediary leverage cycles within a dynamic model of the macroeconomy.[4]

- When no analytical solution is at hand, the numerical techniques required to solve the nonlinear problem are typically simpler for continuous-time methods than for discrete-time counterparts. The reason is that whereas the solution of the discrete-time Bellman equation requires the computation of an expectation, its continuous-time equivalent, the Hamilton-Jacobi-Bellman equation is a deterministic partial differential equation (PDE). This feature has been exploited in macro-finance papers such as Brunnermeier and Sannikov (2014) or He and Krishnamurthy (2013).[5]

- Continuous-time techniques are especially well-suited to analyze situations in which action entails a fixed cost and thus are taken infrequently, the so-called "impulse control" problems, as described in Stokey (2008). For instance, Alvarez, Lippi and Paciello (2011) analyze the price-setting problem of a firm in the presence of both observation and menu costs and Nuño and Thomas (2015) compare alternative monetary policy regimes in an economy with endogeneous sovereign default.[6].

- A case of particular interest is the solution of heterogeneous-agents models. In discrete time, the computation of the aggregate distribution is restricted to the use of numerical techniques (typically Monte Carlo methods). In continuous-time instead there exists a deterministic partial differential equation (the Kolmogorov forward equation), describing the time-varying evolution (law of motion) of the distribution. This simplifies substantially the computation of non-standard models such as Lucas and Moll (2014), the control of a distribution of heterogeneous agents as in Nuño and Moll (2015) or the analysis of games between a large player and a distribution of atomistic agents as in Nuño and Thomas (2016).[7]

The aim of this course is to provide an introduction to continuous-time methods both in theory and in practice, with special emphasis to applications in economics. The course provides the theoretical foundations of stochastic calculus and then introduces the main numerical techniques applied to relevant examples.

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[4] See Adrian, T. and N. Boyarchenko (2013). "Intermediary Leverage Cycles and Financial Stability," Federal Reserve Bank of New York Staff Reports, no. 567.

[5] See Brunnermeier, M. and Y. Sannikov (2014), "A Macroeconomic Model with a Financial Sector," *American Economic Review* 104(2), pp. 379-421; and He, Z., and A. Krishnamurthy (2013), "Intermediary Asset Pricing, *American Economic Review*," 103(2): pp. 732-70.

[6] See Alvarez, F. E., F. Lippi and Luigi Paciello (2011) "Optimal Price Setting With Observation and Menu Costs," *The Quarterly Journal of Economics*, vol. 126(4), pages 1909-1960.

[7] See references below.

Prerequisites

The coursed is mainly aimed at researchers or practitioners in Central Banks, Academia or Investment Banks. No previous exposure to stochastic calculus is required. Participants should have basic knowledge of Calculus, Probability and Economics at a Master or 1^{st} year-PhD level. In addition, participants should have a basic knowledge of programming, especially in Matlab.

Course outline

The course is taught in 4 sessions of 4 hours each. The material will be self-contained.

Lecture 1: Introduction to Stochastic Calculus: Application to Option Pricing.

This lecture will present a concise summary of stochastic calculus that is most useful in economics and finance. We will discuss the properties of the Brownian motion, stochastic integral, Itô's formula and the Kolmogorov forward equation. Then we will apply these techniques to option pricing and we will derive the Black-Scholes formula for European options.

Readings:

- Björk, Tomas (2009). *Arbitrage Theory in Continuous Time*, Oxford University Press. Chapters 4-7.
- Black, Fischer and Myron Scholes (1973). "The Pricing of Options and Corporate Liabilities". *Journal of Political Economy*, 81(3), pp. 637-654.
- Øksendal, Bernt (2007). *Stochastic Differential Equations: An Introduction with Applications*. Springer. Chapters 3-5.
- Shreve, Steven (2013). *Stochastic Calculus for Finance II: Continuous-Time Models,* Springer. Chapter 4.

Lecture 2: Stochastic control: Application to Macro-Finance and Sovereign Default.

This lecture will introduce dynamic programming in continuous-time. We will derive the Hamilton-Jacobi-Bellman (HJB) equation and we will illustrate how to solve it analytically in a model of optimal portfolio selection à la Merton (1969). We will briefly discuss the recent macro-finance literature. The lecture will also analyze problems in which action entails a fixed cost and thus are taken infrequently and will review dynamic games in continuous time, focusing on the concept of Markov Perfect Nash Equilibrium. We will illustrate these concepts in an example: the effects of monetary policy and sovereign default on sovereign debt pricing.

Readings:

- Björk, Tomas (2009). *Arbitrage Theory in Continuous Time*, Oxford University Press. Chapter 19.
- Brunnermeier, Markus K, Thomas Eisenbach, and Yuliy Sannikov (2013). "Macroeconomics With Financial Frictions: A Survey". *Advances In Economics And Econometrics*. Cambridge University Press.
- Merton, R. C. (1969). "Lifetime Portfolio Selection under Uncertainty: the Continuous-Time Case". *The Review of Economics and Statistics* 51 (3): 247–257.
- Øksendal, Bernt (2007). *Stochastic Differential Equations: An Introduction with Applications*. Springer. Chapter 11.
- Stokey, Nancy (2008). *The Economics of Inaction: Stochastic Control Models with Fixed Costs*. Princeton University Press. Chapter 3, 6-9.
- Merton, R. C., 1974. "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates," *Journal of Finance*, 29(2), 449-470.
- Nuño, G. and C. Thomas (2015). "Monetary Policy and Sovereign Debt Vulnerability," mimeo.
- Dixit, A. and R. S. Pindyck (1994), *Investment under Uncertainty*, Princeton University Press.
- Başar, T. and G. J. Olsder (1999), (1999), *Dynamic Noncooperative Game Theory,* Ed. Society for Industrial and Applied Mathematics. 2 edition. Chapters 5-6.

Lecture 3: Numerical techniques.

Most stochastic control problems cannot be solved with pencil and paper. In this lecture we will introduce finite difference methods to solve the HJB equation. We will illustrate them by solving the problem of a household with idiosyncratic risk and borrowing constraints. We will also discuss the application of these techniques to models with fixed costs, as in the case of endogenous default.

Readings:

- Achdou, Y., J.-M. Lasry, P.-L. Lions and B. Moll (2015). "Heterogeneous Agent Models in Continuous Time," mimeo.

- Barles, G. and P. E. Souganidis (1991). "Convergence of Approximation Schemes for Fully Nonlinear Second Order Equations," *J. Asymptotic Analysis*, 4, pp. 271-283.

- Fleming, W. H. and H. M. Soner (2006). *Controlled Markov Processes and Viscosity Solutions*, Springer. Chapter 9.

Lecture 4: Heterogeneous-Agents in Continuous Time and Social Planning and Games with Heterogeneous Agents.

This lecture will discuss how continuous-time models can be applied to different macroeconomic problems. In particular we will focus in heterogeneous-agents economies à la Aiyagari (1994). We will discuss the links between these techniques and the emerging field of "mean-field game theory" in mathematics. Finally, we will discuss the latest advances in the analysis of social optima and games in models with heterogeneous agents. This problems are typically hard to solve as the state distribution is an infinite-dimensional object. We will see how the introduction of calculus in infinite dimensional Hilbert spaces makes the problem tractable and will provide a example in monetary policy.

Readings:

- Achdou, Y., J.-M. Lasry, P.-L. Lions and B. Moll (2015), "Heterogeneous Agent Models in Continuous Time," mimeo.

- Aiyagari, R., (1994), "Uninsured Idiosyncratic Risk and Aggregate Saving," *The Quarterly Journal of Economics*, 109 (3), pp. 659-84.

- Lasry, J.-M. and P.-L. Lions (2007), "Mean Field Games," *Japanese Journal of Mathematics*, 2 (1), pp. 229-260.

- Lucas, R. and B. Moll (2014), "Knowledge Growth and the Allocation of Time," *Journal of Political Economy*, 122 (1), pp. 1-51..

- Nuño, G. and B. Moll (2016). "Controlling a Distribution of Heterogeneous Agents," mimeo.

- Nuño, G. and C. Thomas (2016). "Optimal Monetary Policy with Heterogeneous Agents," mimeo.

- Nuño, G. (2016). "Optimal Social Policies in Mean Field Games," mimeo.

- Dávila, J., J. H. Hong, P. Krusell and J. V. Ríos-Rull (2012) "Constrained Efficiency in the Neoclassical Growth Model With Uninsurable Idiosyncratic Shocks," *Econometrica*, 80(6), pp. 2431-2467.